# A Note on Fractional Coloring and the Integrality gap of LP for Maximum Weight Independent Set

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**Abstract**

We prove a tight connection between two important notions in combinatorial optimization. Let $\mathcal{G}$ be a graph class (i.e. a subset of all graphs) and $r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \frac{\chi_f(G)}{\omega(G)}$ where $\chi_f(G)$ and $\omega(G)$ are the fractional chromatic number and clique number of $G$ respectively. In this note, we prove that $r(\mathcal{G})$ tightly captures the integrality gap of the LP relaxation with clique constraints for the Maximum Weight Independent Set (MWIS) problem. Our proof uses standard applications of multiplicative weight techniques, so it is algorithmic: Any algorithm for rounding the LP can be turned into a fractional coloring algorithm and vice versa. We discuss immediate applications of our results in approximating the fractional chromatic number of certain classes of intersection graphs.

*Keywords:* Fractional coloring, maximum weight independent set, linear programming.

## 1 Introduction

In the Maximum Weight Independent Set Problem (MWIS), we are given graph $G$ and weight function $w : V(G) \to \mathbb{R}_{\geq 0}$. A set $J \subseteq V(G)$ is independent if there is no edge in $J$. Define $w(J) = \sum_{v \in J} w(v)$. Our goal is to compute the maximum weight independent set in $G$. We denote the weight of a maximum

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weight independent set by $\alpha(G, w)$. This problem is cornerstone in combinatorial optimization and has been extensively studied.

We consider the LP relaxation with clique constraints for MWIS. For each vertex $v \in V$, there is a variable $x_v$ indicating whether vertex $v$ is included.

$$\text{(LP)} \quad \max \sum_{v \in V(G)} w(v) x_v$$
$$\text{s.t.} \sum_{v \in C} x_v \leq 1 \text{ for each clique } C \text{ in graph } G$$

In general, the number of cliques can be exponentially large, but for restricted graph classes (e.g. intersection graphs of rectangles in higher dimensional boxes [4,3]), there is only a polynomial number of maximal cliques. Moreover, it is known that all clique constraints are implied by the canonical SDP relaxation of MWIS, as well as the Lovasz theta function [6]. The main question of our interest is:

*How good is (LP) in approximating the maximum-weight independent set?*

For each $G$ and weight function $w$, define $\mathsf{LP}(G, w)$ as the value of an optimal solution for the above LP. The integrality gap $\mathsf{gap}(G, w)$ is the ratio $\frac{\mathsf{LP}(G,w)}{\alpha(G,w)}$.

In this note, we show a tight connection between the integrality gap of (LP) and the fractional chromatic number of a graph. A valid *fractional coloring* for $G$ is a function $\sigma : 2^{V(G)} \rightarrow [0, 1]$ such that (i) the support of $\sigma$ contains only independent sets, and (ii) for each $v \in V(G)$, we have $\sum_{I: v \in I} \sigma(I) \geq 1$. The *fractional chromatic number* $\chi_f(G)$ is defined as the minimum real number $k$ such that there exists a valid fractional coloring $\sigma$, $\sum_I \sigma(I) \leq k$.

For any graph $G$, a *clique replacement operation* on $v$ is performed by creating graph $G' : V(G') = (V(G) \setminus v) \cup \{v_1, \ldots, v_\ell\}$ and $E(G') = E(G \setminus v) \cup \{v_i u : vu \in E(G)\} \cup \{v_i v_j : i, j \in [\ell]\}$. In words, this operation replaces vertex $v$ with a clique $K_\ell$. Let $\mathcal{G}$ be a class of graphs. We say that $\mathcal{G}$ is closed under clique replacement if for any $G \in \mathcal{G}$, a clique replacement operation at $v$ gives us $G' \in \mathcal{G}$. Many natural graph classes are closed under clique replacement, e.g., interval graphs, $d$-dimensional box graphs, disk graphs, and perfect graphs.

**Theorem 1.1** *Let $\mathcal{G}$ be any class of graphs that is closed under clique replacement. The following statements hold:*

- *Suppose that, for any $n$-vertex graph $G \in \mathcal{G}$, we have $\chi_f(G) \leq \gamma(n)\omega(G)$. Then, for any $G \in \mathcal{G}$ and any weight function $w$, we have $\mathsf{LP}(G, w) \leq \gamma(N)\alpha(G, w)$ for some $N$. Moreover, given a fractional coloring with polynomial support, there is a $(1 + \epsilon)\gamma(N)$ approximation for MWIS via rounding (LP), for $N = O(n^2/\epsilon)$.*
- *Assume $\mathsf{LP}(G, w) \leq \gamma(n)\alpha(G, w)$ for all $w$. Then we have $\chi_f(G) \leq \gamma(n)\omega(G)$. Moreover, given a polynomial-time $\gamma(n)$-approximation LP rounding algorithm for MWIS, we can efficiently compute a fractional coloring using at most $(1+$*

$\epsilon)\gamma(n)\omega(G)$ *colors for any* $\epsilon > 0$.

The gap between $\chi(G)$ and $\omega(G)$ has received a lot of attention in the context of intersection graphs. In particular, many old problems in mathematics are related to $\chi$-boundedness[3] of intersection graphs (see for instance [2,5] and references therein). We hope that this work will encourage the study of $\chi_f(G)/\omega(G)$. Our results have many immediate applications, giving both new algorithmic and integrality gap results. Due to the space limit, we omit the applications.

## 2 The Equivalence

**Fractional Coloring $\Longrightarrow$ LP Gap:** Consider any graph $G = (V, E)$, $n = |V|$, and $G \in \mathcal{G}$. We will show that $\alpha(G, w) \geq \mathsf{LP}(G, w)/\gamma(n)$.

Let $x$ be an optimal LP solution for (LP). First, assume that $x_v$ is in an integral multiple of $1/q$ for some integers $q$. By standard LP theory, this is possible. Let $x_v = q_v/q$. We create a graph $G'$ from $G$ as follows: For each vertex $v \in V(G)$, perform a clique replacement operation on $v$ by replacing $v$ with a clique $X_v$ of size $q_v$. Observe that $\omega(G') \leq q$: Let $C'$ be a clique in $G'$. Consider the set $C = \{v \in V(G) : X_v \cap C' \neq \emptyset\}$. The LP constraint guarantees that $\sum_{v \in C} x_v \leq 1$ and therefore $|C'| \leq \sum_{v \in C} |X_v| = \sum_{v \in C} q_v \leq q$.

Since $\mathcal{G}$ is closed under clique replacement operation, we have $G' \in \mathcal{G}$ and that $\chi_f(G') \leq \gamma(N)q$. Let $\sigma$ be an optimal fractional coloring of $G'$. We sample an independent set $J$ where each $J \subseteq V(G')$ is sampled with probability $\sigma(J)/\chi_f(G)$. Therefore, each vertex $v \in V(G')$ is sampled with probability $\sum_{I: v \in I} \sigma(I) \geq 1/\chi_f(G')$. So we get an independent set $J$: $\mathbb{E}[w(J)] = \sum_{v \in V(G')} w(v) Pr[v \in J] \geq \frac{1}{\chi_f(G')} \sum_{v \in V(G')} w(v)$. This is at least $\frac{1}{\gamma(N)q} \sum_{v \in V(G)} w(v) q_v = \mathsf{LP}(G, w)/\gamma(N)$.

This concludes the proof. Remark that $N$ can be very large compared to $n$, but this does not affect the ratio if $\gamma$ is a constant function. If $\gamma$ is not a constant function, we can reduce the value of $N$ to $O(n^2/\epsilon)$, while preserving the ratio within a factor of $(1 + \epsilon)$. The proof is omitted, due to space limitation.

**LP Gap $\Longrightarrow$ Coloring:** Let $G$ be a graph on $n$ vertices. If $\mathsf{gap}(G, w) \leq \gamma(n)$ for all weight vectors $w$, then $\chi_f(G) \leq \gamma(n)\omega(G)$. Moreover, we show how to compute a fractional coloring using at most $(1 + \epsilon)\gamma(n)\omega(G)$ colors for any $\epsilon > 0$.

The following linear constraints check whether the graph is $1/\eta$-colorable.

$$\text{(P)} \quad \sum_{I: v \in I} \sigma(I) \geq \eta \text{ for all } v \in V(G)$$
$$\sum_{I} \sigma(I) \leq 1$$

[3] A graph is $\chi$-bounded if $\chi(G) \leq f(\omega(G))$ for some function $f$.

Our goal is to find a feasible solution $\sigma$ that satisfies every constraint. Applying a standard multiplicative weight framework, our algorithm does the following steps:

(i) Start with initial weight function $w^{(1)}$ where $w_v^{(1)} = 1$ for all $v$.

(ii) In iteration $t$, compute a solution $\sigma^{(t)}$ that satisfies the "weighted average constraint" $\sum_v w_v^{(t)}(\sum_{I:v\in I} \sigma(I) - \eta) \geq 0$.

(iii) Update the weight $w^{(t)}$ to $w^{(t+1)}$. Then return to Step (ii).

**Theorem 2.1** *[1] There is an update strategy such that, after $T$ rounds, solution $\sigma = \frac{1}{T}\sum_{t=1}^{T} \sigma^{(t)}$ $(1-\epsilon)$-satisfies all constraints, i.e. for all $v$, $\sum_{I:v\in I} \sigma(I) \geq (1-\epsilon)\eta$.*

It only remains to show that we can compute a solution that satisfies the "weighted average constraint", which means finding $I$ with $w(I) \geq \eta\, w^{(t)}(V)$ on $(G, w^{(t)})$. Consider the linear program for MWIS, (LP), using weights $\{w_v^{(t)}\}_{v\in V}$. We obtain a fractional solution $x$ with weight $\frac{1}{\omega(G)} \sum_{v\in V} w_v^{(t)}$ by setting $x_v = 1/\omega(G)$ for all $v \in V$. Since $\sum_{v\in C} x_v = |C|/|\omega(G)| \leq 1$, for every clique $C$, it is clear that this is a solution to the LP. This implies that there is an integer solution with weight $\frac{1}{\gamma(n)\omega(G)} \sum_{v\in V} w_v^{(t)} = \eta \sum_{v\in V} w_v^{(t)}$, that is, there is an independent set $I'$ with the desired weight. Furthermore, we use a $\gamma(n)$-approximation LP rounding algorithm to find $I'$ of total weight $\frac{1}{\gamma(n)}\sum_v w_v^{(t)} x_v = w^{(t)}(V)/\omega(G)\gamma(n)$.

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