Hedonic Coalition Formation in Networks

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Abstract

Coalition formation is a fundamental problem in the organization of many multi-agent systems. In large populations, the formation of coalitions is often restricted by structural visibility and locality constraints under which agents can reorganize. We capture and study this aspect using a novel network-based model for dynamic locality within the popular framework of hedonic coalition formation games. We analyze the effects of network-based visibility and structure on the convergence of coalition formation processes to stable states. Our main result is a tight characterization of the structures based on which dynamic coalition formation can stabilize quickly. Maybe surprisingly, polynomial-time convergence can be achieved if and only if coalition formation is based on complete or star graphs.

Introduction

Coalition formation is an essential process for exploiting synergies in multi-agent systems. A natural and versatile model for studying formal aspects of coalition formation are hedonic games (Dreze and Greenberg 1980). In these games the outcome of strategic interaction is a coalition structure, i.e., a partition of the agent set. A central feature of hedonic games is that the payoff for each agent is solely determined by the set of agents in his coalition, regardless of which other coalitions may or may not be present. Hedonic games capture many of the challenges of coalition formation, and various aspects of different stability concepts have been studied in recent years (see e.g., (Banerjee, Konishi, and Sönmez 2001; Bogomolnaia and Jackson 2002; Aziz, Brandt, and Seedig 2011; Aziz, Brandt, and Harrenstein 2014)).

In the standard model of hedonic games, there is no a priori restriction on the coalitions that the agents can form (except agent preference). However, in many large-scale systems, the formation of coalitions is often restricted by additional structural visibility and locality constraints under which agents can (re-)organize. As an example, consider scientific publishing as a coalition formation process, where researchers group themselves into teams working on projects and papers. Here knowledge and visibility is a fundamental challenge – one would not expect researchers to group with any other colleagues instantaneously. Instead, agents have to learn and get to know about each other before they can engage in a joint project. For systems with computational agents, locality can also be a necessity since agents might be physically or computationally incapable of exploring all opportunities for coalition formation. Naturally, locality can have a severe impact on various aspects of coalition formation, and many of the interesting consequences are not well-understood from a formal point of view.

In this paper, we incorporate the aspect of limited visibility into hedonic games and study the impact of coalition discovery on dynamic formation processes. Our model is similar to recent graph-based models for locality in stable matching. In *locally stable matching* (Arcaute and Vassilvitskii 2009) a set V of agents strives to match into pairs. Agents are nodes of an underlying network $N = (V, L)$ with permanent links L. These links express permanent visibility that grounds in, e.g., spatial closeness, family or co-worker relations, etc., and they are unaffected by coalition formation. In addition, the current matching introduces temporary links into the network. Pairs of agents become accessible if their distance in the current network is low, and they might deviate if this is compatible with their preferences. After deviation the set of temporary links changes. In this way, agents are subject to changing opportunities for partnerships during the dynamics.

We model locality issues in coalition formation beyond pairs and analyze a natural extension of the matching model outlined above. In addition to the network $N = (V, L)$ of permanent links, for each existing coalition we insert a clique of temporary links into the network, i.e., within a coalition each pair of agents becomes connected and they stay connected for as long as the coalition exists. If a coalition is abandoned, the agents lose the temporary connections provided through the common coalition (unless they are contained in L). To discover and form a new coalition, agents have to re-organize using the current network of permanent and temporary contacts. Depending on the application, it might not be necessary for every agent to be in contact with every other agent of the new coalition beforehand (consider, e.g., the formation of a program committee for a scientific conference). In fact, a single agent being in contact with all others, or short pairwise distances, or even just connectivity within the group might be enough. Hence, given the current network, we assume a coalition C is available for deviation if it has an inherent structure, which we term *formation graph* of C. We will analyze how the structures of formation graphs influence the convergence time of the coalition formation process. Before we state our results and connections to related work, let us formally introduce the model.

Preliminaries

A *coalition formation game* consists of a set V of agents, and a set $\mathcal{C} \subseteq 2^V$ of *possible coalitions*, where 2^V denotes the power set of V. Observe that $\mathcal{C} = 2^V$ is a special case, and in general the set of possible coalitions can be restricted. We denote the number of agents by $n = |V|$ and the number of possible coalitions by $m = |\mathcal{C}|$. A *state* is a *coalition structure* $\mathcal{S} \subseteq \mathcal{C}$ such that for each $v \in V$ we have $|\{C \mid C \in \mathcal{S}, v \in C\}| \leq 1$. That is, each agent is involved in at most one coalition. Each coalition C has a weight or benefit $w(C) > 0$, which is the profit given to each agent $v \in C$. For a coalition structure $\mathcal{S}$, a *blocking coalition* is a coalition $C \in \mathcal{C} \setminus \mathcal{S}$ with $w(C) > w(C_v)$ where $v \in C_v \in \mathcal{S}$ for every $v \in C$ which is part of a coalition in $\mathcal{S}$, that is, each vertex $v \in C$ is either not in a coalition or C has a larger benefit than its current coalition. An *improvement step* is the act of adding such a blocking coalition to the state $\mathcal{S}$, while at the same time removing all conflicting coalitions. We also call this the *resolution of a blocking coalition*. A stable state or *stable coalition structure* $\mathcal{S}$ does not have any blocking coalitions.

For a *local coalition formation game* we also have a network $N = (V, L)$ with a set of *permanent links* L that models visibility between players and a set $\mathcal{G}$ of graphs used as deviation structures. A graph $G \in \mathcal{G}$ is denoted a *formation graph*. In general, $\mathcal{G}$ can contain more than one formation graph. For any coalition C, we denote by K_C the clique between the agents of C. A coalition $C \in \mathcal{C}$ is called *accessible* in state $\mathcal{S}$ if the vertices in C, along with some of the edges in L and the current coalitions, form a graph isomorphic to some graph in $\mathcal{G}$. Formally, $C \in \mathcal{C}$ is accessible in $\mathcal{S}$ if there is some bijective map from the vertices of some $G = (V_G, E_G) \in \mathcal{G}$ to the players in C such that for every $e \in E_G$ the corresponding edge for C is in $L \cup \bigcup_{C' \in \mathcal{S}} K_{C'}$. We slightly abuse notation and denote this situation by $G(C) \in N \cup \mathcal{S}$. A state $\mathcal{S}$ has a *local blocking coalition* $C \in \mathcal{C}$ if C is a blocking coalition and C is accessible. Consequently, a *locally stable coalition structure* is a coalition structure without local blocking coalitions. A *local improvement step* is the resolution of such a local blocking coalition, that is, the blocking coalition is added to $\mathcal{S}$ and all conflicting coalitions are removed.

To visualize the dynamics of local improvement steps consider the example displayed in Fig. 1.

Results

In our games it is straightforward to observe that a lexicographic potential function exists (Abraham et al. 2008; Mathieu 2010) and every sequence of improvement steps leads into a stable state. This holds in particular for sequences of local improvement steps. Theorem 1 below shows that if every formation graph $G \in \mathcal{G}$ is a clique, then there are always short "paths to stability", i.e., polynomial-time sequences of improvement steps to stable states. In this case, we obtain convergence in time $O(mn)$ even for random dynamics, where in each round the deviating coalition is chosen uniformly at random. Theorem 2 proves the existence of short paths to stability if every formation graph is a star. In contrast, for every other graph structure G we provide in Theorem 3 an instance with n agents, $\mathcal{G} = \{G\}$, $m = \Theta(n)$ possible coalitions and an initial state such that the unique sequence of improvement steps based on formation graph G has length $2^{\Theta(n)}$.

This provides a tight characterization of graph structures for visibility based on which fast convergence can be achieved. In particular, for cliques we can obtain fast convergence for random dynamics. This even holds if $\mathcal{G}$ consists of a single or multiple cliques. For cliques and stars we can always guarantee the existence of a short path to stability, which again holds if $\mathcal{G}$ has one or multiple graphs (which are either all cliques or all stars). In contrast, for any other graph structure G, there are games where every path must be of exponential length, even if $\mathcal{G}$ contains only this one graph G. This provides the first insights on how dynamic network structure affects coalition formation processes beyond the special case of matching. Finally, we also discuss the relations of our model to locally stable matching, variations of our visibility concept, and possible directions for future work.

Our results are strongest when agents strive to form coalitions of small size. The set $\mathcal{C}$ specifies the allowed coalitions, and if $m = |\mathcal{C}| \in \text{poly}(n)$, then many involved problems (finding a local blocking coalition, or the one with maxi-

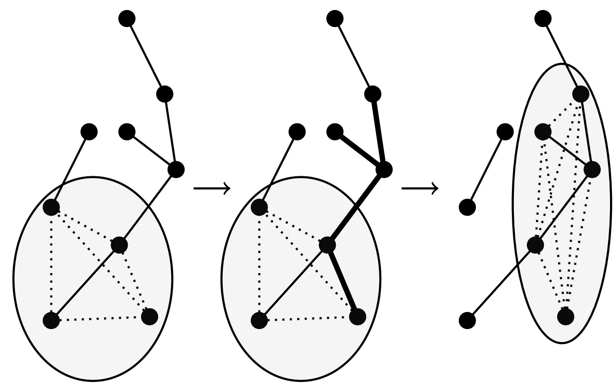

Figure 1: Display of a local improvement step. Agents are depicted by vertices, coalitions by an ellipse covering involved agents. Thin edges indicate permanent links, dotted edges depict temporary links generated by some coalition. The set of formation graphs $\mathcal{G}$ is all graphs of diameter at most 3. Left part: Initial state with a coalition of size 4. Middle part: Thick edges indicate a map of some formation graph that can be used to discover a new coalition. Note that the graph is composed of permanent as well as temporary links. Right part: Outcome once the new coalition is formed including the change in temporary links.

mum profit) can be solved trivially in time $\text{poly}(n)$. In particular, if the formation graphs are of constant size, then we can assume $m \in \text{poly}(n)$, since only coalitions of constant size can form. Under this assumption, we can obtain paths of length $\text{poly}(n)$ for cliques and stars. In contrast, our exponential lower bound already holds when, e.g., the formation graph is a path with 4 nodes.

Related Work

We extend the popular framework of hedonic games, a central concept for modeling coalition formation in multi-agent systems, game theory, and algorithms. There is a vast literature on hedonic coalition formation, starting with (Dreze and Greenberg 1980). Recently, the existence and computational complexity of computing stable states has attracted some interest (see, e.g., the works of Hajduková (2006); Cechlárova (2008); Gairing and Savani (2010); Sung and Dimitrov (2010)). In addition, various stability concepts – many focusing on the core of the game – have been studied (Banerjee, Konishi, and Sönmez 2001; Bogomolnaia and Jackson 2002; Aziz, Brandt, and Harrenstein 2014). A central issue is the way payoff is derived from the current state of the game. In additively separable hedonic games each agent assigns a value to each agent individually and then sums up over the agents in his coalition (Banerjee, Konishi, and Sönmez 2001; Burani and Zwicker 2003; Aziz, Brandt, and Seedig 2011). A similar approach can be found in fractional hedonic games where instead of the sum the average over all agents in the coalition is taken (Aziz, Brandt, and Harrenstein 2014). Here we focus on convergence and study the case of correlated payoffs. Correlated preferences have proven useful to guarantee paths to stability in various settings (e.g., (Mathieu 2010)).

The idea to use an underlying network structure to derive properties of the game has recently found a lot of interest in various disciplines. In many settings, the network is used to determine payoffs, or even network creation is considered as strategic process (Jackson 2008). In contrast, we use the network mainly to determine visibility and availability. Our approach to locality is related to existing graph-based models for limited interaction. Most prominently, there is a large body of works treating the Myerson value in graph-based cooperative games (Myerson 1977; Owen 1986). Solution concepts in graph-based cooperative games have received a lot of interest over the last decades in game theory and economics. For example, there has been recent interest in the cost of stability based on the graph structures underlying the game (Meir et al. 2013).

Myerson's idea can be seen as a static variant of our model – it excludes the formation of coalitions that are not connected in the underlying network. We obtain a similar scenario if the class of formation graphs is the set of all connected graphs and we refrain from introducing temporary links into the network. The latter, however, are the key aspect for dynamic discovery of new coalitions, which is the distinguishing feature we capture with our approach.

Our approach can be regarded as an extension of locally stable matching (Arcaute and Vassilvitskii 2009). Locally stable matching and related variants have attracted recent interest, especially approximation algorithms for finding a stable matching of maximum size (Askalidis et al. 2013; Cheng and McDermid 2013) or convergence properties of dynamics (Hoefer 2013; Hoefer and Wagner 2013; 2014). A similar idea of limited visibility has recently been studied in the context of strategic network creation (Bilò et al. 2014).

Our convergence results rely on agent preferences being correlated using a single weight $w(C)$ for each coalition $C \in \mathcal{C}$. This assumption is natural when agents share benefits of a coalition equally, such as, in scientific publishing with alphabetical ordering of authors. Clearly, there are other cases where profit is not divided equally, and there is a huge body of work concerned with stability in sharing and division problems. For non-equal sharing, when each agent has an arbitrary preference order over coalitions of $\mathcal{C}$, improvement dynamics do not converge at all since stable states can be absent, even for special cases and without locality restrictions (Aziz and Brandl 2012). Obviously, locality restrictions make matters even harder, and we expect strong impossibility results for the case of general preferences, similar to locally stable matchings (Hoefer and Wagner 2013).

Paths to Stability

We are interested in finding sequences of local improvement steps which lead to a stable state and only need a small number of steps. Such sequences guarantee that from every initial state the system can quickly be brought to a stable state, where all agents are content with their current situation considering their current alternatives. We will analyze the existence of short paths to stability depending on the type of formation graphs for visibility. It turns out that for two very natural choices – the clique, where all agents have to know each other to form a coalition, and the star, where one central agent knows all other agents – we obtain positive results. In contrast, all other graph structures have properties that can be exploited to construct instances where all paths to stability are of exponential length.

Cliques

Theorem 1. *Let $\mathcal{G} \subseteq \{G_i | i = 1 \dots n\}$ where G_i denotes a clique of size i. Then every local coalition formation game using $\mathcal{G}$ has a path to stability of length at most n using only local improvement steps from any starting state $\mathcal{S}$. Furthermore, random dynamics converge to a stable state in expected time at most $O(mn)$.*

Proof. We observe that the set of accessible coalitions only can shrink in the course of the dynamics. A coalition is only accessible if all pairs of involved agents are connected through L or through a temporary link due to being currently in the same coalition. Thus, creation of a coalition by resolving some local blocking coalition does not introduce any new additional links to the network. On the other hand some links can get lost if overlapping coalitions are removed. Hence, improvement steps only shrink the set of temporary links and, likewise, the set of accessible coalitions.

Assume that C is a local blocking coalition of maximal benefit among all local blocking coalitions in $\mathcal{S}$. Once C is formed, it will not be removed by any other coalition created through subsequent improvement steps as no new coalitions of higher value will be discovered and become accessible. Thus, repeatedly resolving the most valuable local blocking coalition results in a stable state after at most n steps.

If instead of picking the most valuable we pick a local blocking coalition at random, then with probability at least $\frac{1}{m}$ we pick one of the most valuable local blocking coalitions. Since these coalitions are never removed, in expectation after at most mn steps we reach a stable state. □

Stars

By H_i we denote a star consisting of a center and $i-1$ leaves. We analyze the case where $\mathcal{G} \subseteq \{H_i | i = 1 \dots n\}$ using *coalition formation games with constraints* introduced by Hoefer and Wagner (2014). In this framework, for each state $\mathcal{S}$ we consider two sets of rules – *generation rules* that determine candidate coalitions, and *domination rules* that forbid some of the candidate coalitions. The set of undominated candidate coalitions then forms the blocking coalitions for state $\mathcal{S}$.

More formally, there is a set $T \subseteq \{(\mathcal{T}, C) \mid \mathcal{T} \subset \mathcal{C}, C \in \mathcal{C}\}$ of *generation rules*. If in the current state $\mathcal{S}$ we have $\mathcal{T} \subseteq \mathcal{S}$ and $C \notin \mathcal{S}$, then C becomes a candidate coalition. In addition, there is a set $D \subseteq \{(\mathcal{T}, C) \mid \mathcal{T} \subset \mathcal{C}, C \in \mathcal{C}\}$ of *domination rules*. If $\mathcal{T} \subseteq \mathcal{S}$ for the current state $\mathcal{S}$, then C cannot be in $\mathcal{S}$. In particular, if C exists and the last missing coalition of $\mathcal{T}$ is formed, then C has to be dropped.

The undominated candidate coalitions represent the blocking coalitions for $\mathcal{S}$. A coalition structure is stable if the set of blocking coalitions is empty.

The generation rules of a coalition formation game with constraints are called *consistent* if $T \subseteq \{(\{C_1\}, C_2) \mid C_1 \cap C_2 \neq \emptyset\}$, that is, all generation rules have only a single coalition in their precondition and the candidate coalition shares at least one agent. The domination rules of a coalition formation game with constraints are called *consistent* if $D \subseteq \{(\mathcal{S}, C) \mid \mathcal{S} \subset \mathcal{C}, C \in \mathcal{C}, C \notin \mathcal{S}, \exists S \in \mathcal{S} : S \cap C \neq \emptyset\}$, that is, at least one of the coalitions in $\mathcal{S}$ overlaps with the dominated coalition. Hoefer and Wagner show (2014) that every coalition formation game with constraints and consistent generation and domination rules has a path to stability of length $O(nm^2)$ from any starting state.

Theorem 2. *Let $\mathcal{G} \subseteq \{H_i | i = 1 \dots n\}$ where H_i denotes a star consisting of a center and $i-1$ leaves. Then every local coalition formation game using $\mathcal{G}$ can be formulated as a coalition formation game with constraints and consistent generation and domination rules. Therefore, in every such game there is a path to stability of length at most $O(nm^2)$ using only local improvement steps from any starting state $\mathcal{S}$.*

Proof. To embed our games into the framework, we first need to express our star-based visibility constraints using consistent generation and domination rules. The set of domination rules is quite easy to define as the only reason an accessible coalition is not formed is because one of the involved players is already part of a better or equally profitable coalition. Thus we set

$$D = \{(\{C'\}, C) \mid C, C' \in \mathcal{C}, C \cap C' \neq \emptyset, w(C') \geq w(C)\}.$$

For the generation rules we show that in its formation graph no accessible coalition relies on temporary links of more than one existing coalition. Assume for contradiction that in the formation graph of accessible coalition C we rely on temporary links of two coalitions $C_1, C_2 \in \mathcal{S}$. In the star all edges share the center vertex, so this vertex must be part of both C_1 and C_2, a contradiction to $C_1, C_2 \in \mathcal{S}$. Thus, for the generation rules we have

$$\begin{aligned} T = &\{(\emptyset, C) \mid C \in \mathcal{C}, H_{|C|} \in N, H_{|C|} \in \mathcal{G}\} \\ &\cup \{(\{C'\}, C) \mid C, C' \in \mathcal{C}, C \cap C' \neq \emptyset, \\ &\quad H_{|C|} \in N \cup \{C'\}, H_{|C|} \in \mathcal{G}\}. \end{aligned}$$

Hence, we have derived consistent generation and domination rules. We now have to show that the blocking dynamics are implemented correctly.

First assume that C is a local blocking coalition for $\mathcal{S}$. Then C is accessible, that is, $H_{|C|} \in N \cup \mathcal{S}$ for some $H_{|C|} \in \mathcal{G}$. As discussed above temporary links of at most one coalition are used in $H_{|C|}$. Thus either C is always accessible through N or there exists some $C' \in \mathcal{S}$ such that $H_{|C|} \in N \cup \{C'\}$. Then $(\{C'\}, C) \in T$ which makes C a candidate coalition in $\mathcal{S}$. Furthermore, for C to be a local blocking coalition it also has to be a (normal) blocking coalition, i.e., there is no coalition $C' \in \mathcal{S}$ such that $C \cap C' \neq \emptyset$ and $w(C') \geq w(C)$. Consequently, the candidate coalition C is undominated in $\mathcal{S}$ which makes C a blocking coalition in $\mathcal{S}$ for the coalition formation game with constraints.

Conversely, let C be a blocking coalition in $\mathcal{S}$ for the coalition formation game with constraints. Then C is undominated, that is, C is a blocking coalition. Furthermore, C is a candidate coalition. Thus, C is accessible in $\mathcal{S}$. In consequence, C is a local blocking coalition for $\mathcal{S}$.

Now resolving the local blocking coalition C results in deleting all overlapping coalitions. By definition of blocking coalitions all those existing overlapping coalitions are of smaller value then C. Similarly, resolving the undominated candidate coalition C results in deleting all coalitions dominated by C which by definition are exactly those coalitions overlapping with C and of less or equal value than C. As C was undominated, all those coalitions have to be of smaller value than C. Thus, the set of deleted coalitions coincides in both cases. This proves the theorem. □

Other Graph Structures

The following structural lemma is a key insight that we will use for the proof of the general lower bound in this section.

Lemma 1. *Let $G = (V, E)$ be some arbitrary undirected connected graph. If for every simple path $v_1v_2v_3v_4$ with (v_1, v_2), (v_2, v_3), $(v_3, v_4) \in E$ there also exists an edge $(v_1, v_3) \in E$, then G is either a clique or a star.*

Proof. If G does not have a simple path of at least 3 edges, then G is a star.

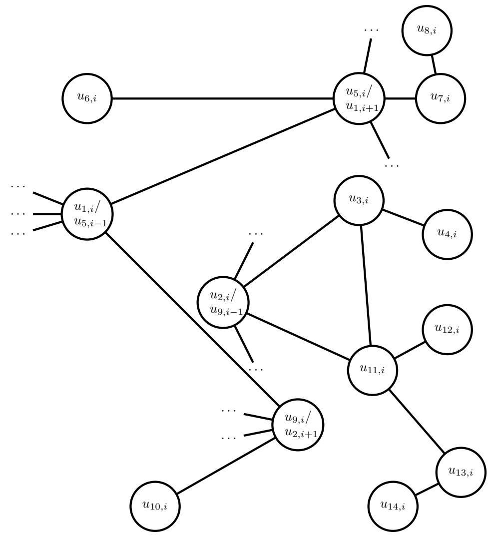

Figure 2: Social network of gadget i

Otherwise, assume G holds some simple path $v_1v_2v_3v_4$ with $(v_1, v_2), (v_2, v_3), (v_3, v_4) \in E$. Suppose that for every such path we have also $(v_1, v_3) \in E$. Then considering this path labeled in forward and backward direction we know that $(v_1, v_3) \in E$ implies $(v_2, v_4) \in E$. Thus we also have the simple path $v_1v_2v_4v_3$ and conclude $(v_1, v_4) \in E$. This means v_1, v_2, v_3 and v_4 form a clique C_1 in G. If $G \neq C_1$, then by connectivity there exists at least one edge e connecting C_1 to $V \setminus C_1$. Let v_5 be the vertex in $e \cap (V \setminus \{v_1, v_2, v_3, v_4\})$ and w.l.o.g. let $e = \{v_1, v_5\}$. Using the edges in C_1 we have the paths $v_5v_1v_2v_3$, $v_5v_1v_3v_2$, and $v_5v_1v_4v_2$ and thus also edges from v_5 to all other vertices in C_1. Hence v_1, v_2, v_3, v_4 and v_5 form a clique C_2 in G. Now we can inductively apply the same arguments until each vertex of V is included. Thus G needs to be a clique. □

Theorem 3. *Let $G = (V, E)$ be an arbitrary connected graph of constant size which is neither a clique nor a star. For every $n \in \mathbb{N}$ there is a local coalition formation game with n agents, $m = \Theta(n)$ coalitions, $\mathcal{G} = \{G\}$ and an initial state such that every sequence to a stable state requires $2^{\Theta(n)}$ improvement steps.*

Proof.

By Lemma 1, we know that there are vertices $v_1, v_2, v_3, v_4 \in V$ such that (v_1, v_2), (v_2, v_3), $(v_3, v_4) \in E$ but $(v_1, v_3) \notin E$. Let $G_{rest} = (V_{rest} = V \setminus \{v_1, v_2, v_3, v_4\}, E_{rest} = E \setminus \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$. We will show how to construct a network of permanent links involving only the vertices v_1, v_2, v_3, and v_4 as well as (v_1, v_2), (v_2, v_3), and (v_3, v_4) for each coalition. For each coalition C, all other vertices V_{rest} are unique (denoted by $V_{C,rest}$) and are connected by links according to E_{rest} (denoted by $E_{C,rest}$) with each other as well as with v_1, v_2, v_3 and v_4. The network is composed of a row of identical gadgets which each have a *starting link* $\{u_{1,i}, u_{2,i}\}$ and two *final links* $\{u_{3,i}, u_{5,i}\}$ and $\{u_{9,i}, u_{11,i}\}$. Those links are only

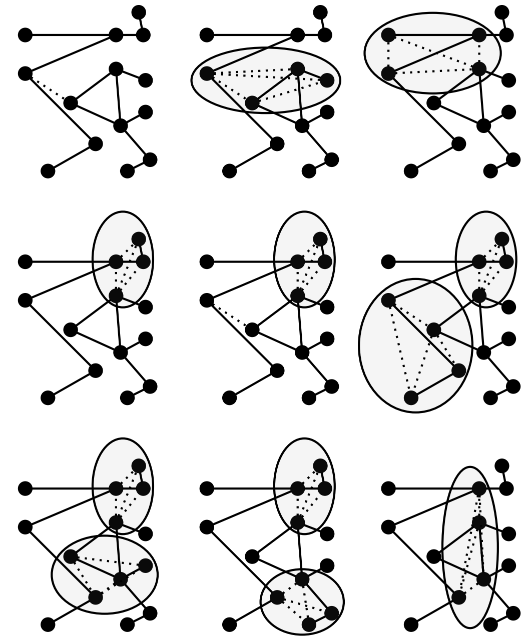

Figure 3: Dynamics of gadget i: The vertices are in the same order as in Figure 2. In step 1 and step 5 the starting link provided by gadget $i-1$ is used. For the other steps temporary links from the coalitions existing in gadget i are used for the improvement steps. In the last figure gadget i is in the right state to provide the starting link for gadget $i+1$.

temporary, that is, they are only available when their incident vertices are in the same coalition. The dynamics are designed such that the starting link is needed once to create the first final link and a second time to create the second final link. The coalition providing the starting link of gadget $i+1$ can only be created when both final links of gadget i are available. Thus to create the coalition which provides the final link of gadget k we need the starting link of gadget 1 2^k times, that is, coalition C_0 has to be formed 2^k times.

For gadget N_i we have 7 coalitions $C_{1,i}, \dots, C_{7,i}$. The vertex set of N_i consists of $\{u_{1,i}, \dots, u_{14,i}\} \cup \bigcup_{j=1}^{7} V_{C_{j,i},rest}$ and the link set of $\{\{u_{1,i}, u_{5,i}\}$, $\{u_{1,i}, u_{9,i}\}$, $\{u_{2,i}, u_{3,i}\}$, $\{u_{2,i}, u_{11,i}\}$, $\{u_{3,i}, u_{4,i}\}$, $\{u_{3,i}, u_{11,i}\}$, $\{u_{5,i}, u_{6,i}\}$, $\{u_{5,i}, u_{7,i}\}$, $\{u_{7,i}, u_{8,i}\}$, $\{u_{9,i}, u_{10,i}\}$, $\{u_{11,i}, u_{12,i}\}$, $\{u_{11,i}, u_{13,i}\}$, $\{u_{13,i}, u_{14,i}\}\}$ $\cup \bigcup_{j=1}^{7} E_{C_{j,i},rest}$. Gadget 1 additionally holds players w_1 and w_2 with links $\{w_1, u_{1,1}\}$, $\{w_1, u_{2,1}\}$, and $\{w_2, u_{2,1}\}$ and a coalition $C_0 = \{w_1, w_2, u_{1,1}, u_{2,1}\} \cup V_{C_0,rest}$ of value 1. The transition from gadget i to gadget $i+1$ is realized by associating $u_{5,i}$ with $u_{1,i+1}$ and $u_{9,i}$ with $u_{2,i+1}$. The social network of some gadget i is illustrated in Figure 2. The gadget vertices are distributed among the coalitions as follows. $C_{1,i} = \{u_{1,i}, u_{2,i}, u_{3,i}, u_{4,i}\} \cup V_{C_{1,i},rest}$ with a value of $4i+1$, $C_{2,i} = \{u_{1,i}, u_{3,i}, u_{5,i}, u_{6,i}\} \cup V_{C_{2,i},rest}$ with a value of $4i+2$, $C_{3,i} = \{u_{3,i}, u_{5,i}, u_{7,i}, u_{8,i}\} \cup V_{C_{3,i},rest}$ with a value of $4i+3$, $C_{4,i} = \{u_{1,i}, u_{2,i}, u_{9,i}, u_{10,i}\} \cup V_{C_{4,i},rest}$ with a value of $4i+1$, $C_{5,i} = \{u_{2,i}, u_{9,i}, u_{11,i}, u_{12,i}\} \cup$

$V_{C_{5,i},rest}$ with a value of $4i + 2$, $C_{6,i} = \{u_{9,i}, u_{11,i}, u_{13,i}, u_{14,i}\} \cup V_{C_{6,i},rest}$ with a value of $4i + 3$, and $C_{7,i} = \{u_{3,i}, u_{5,i}, u_{9,i}, u_{11,i}\} \cup V_{C_{7,i},rest}$ with a value of $4i + 4$. For the initial state we choose $\mathcal{S}_0 = \emptyset$.

We will describe the dynamics of gadget 1 as well as the connection to the next gadget. The other gadgets work similarly. If no player of gadget 1 is involved in any coalition, the only accessible coalition is C_0. Once C_0 is formed, $C_{1,1}$ and $C_{4,1}$ become accessible because the starting link $\{u_{1,1}, u_{2,1}\}$ becomes available. As both coalitions are of higher value than C_0, they are local blocking coalitions. W.l.o.g. assume that $C_{1,1}$ is formed first. Now $C_{4,1}$ is not a blocking coalition any more but the temporary link $\{u_{1,1}, u_{3,1}\}$ becomes available which makes $C_{2,1}$ a local blocking coalition. Forming $C_{2,1}$ results in $u_{2,1}$ being single again. Additionally, the first final temporary link $\{u_{3,1}, u_{5,1}\}$ exists and $C_{3,1}$ becomes a local blocking coalition. Now forming $C_{3,1}$ player $u_{1,1}$ becomes single again, that is, C_0 (which is always accessible) is a blocking coalition again. Further $\{u_{3,1}, u_{5,1}\}$ is still present. Now the only available local blocking coalition again is C_0. This time after C_0 is formed $C_{1,1}$ is not a blocking coalition because $u_{3,1}$ is involved in a better coalition. Thus the only option now is the local blocking coalition $C_{4,1}$. As above, after forming $C_{4,1}$, $C_{5,1}$ becomes available. Through $C_{5,1}$ we then get the second final link $\{u_{9,1}, u_{11,1}\}$. Now we can either directly form $C_{7,1}$ or first $C_{6,1}$ and then $C_{7,1}$ (destroying $C_{6,1}$ again). Note that $C_{6,1}$ is necessary to make C_0 available again for the case we would have chosen $C_{4,1}$ instead of $C_{1,1}$ first. The existence of $C_{7,1}$ provides the temporary link $\{u_{5,1}, u_{9,1}\} = \{u_{1,2}, u_{2,2}\}$ for the fist time. This initiates the same dynamics in the second gadget. Observe that the dynamics cannot terminate prematurely by resolving local blocking coalitions in a different order. Thus, in the end $C_{7,k}$ is necessary for the coalition structure to be stable and 2^k creations of C_0 are necessary to form $C_{7,k}$.

For better understanding a schematic illustration of the dynamics taking place in gadget i is displayed in Figure 3. In particular, one can observe how the starting link has to be provided by gadget $i - 1$ twice to allow gadget i to provide the starting link for gadget $i + 1$.

Observe that we heavily rely on temporary links being temporary and not part of $E_{C,rest}$. As G holds a path of 3 edges with no connection between the first and the third vertex, we can actually arrange $E_{C,rest}$ such that the 'right' links are missing when forming a coalition and only be inserted by this exact coalition. However, the structure of G might require an edge between the first and the fourth and/or the second and the fourth vertex of the path, that is, two vertices which are part of the shared gadget and not the part which is unique for each coalition anyway. Luckily these edges – namely $\{u_{1,i}, u_{4,i}\}$ and $\{u_{2,i}, u_{4,i}\}$ for $C_{1,i}$, $\{u_{1,i}, u_{6,i}\}$ and $\{u_{3,i}, u_{6,i}\}$ for $C_{2,i}$, $\{u_{3,i}, u_{7,i}\}$ and $\{u_{3,i}, u_{8,i}\}$ for $C_{3,i}$, $\{u_{1,i}, u_{10,i}\}$ and $\{u_{2,i}, u_{10,i}\}$ for $C_{4,i}$, $\{u_{2,i}, u_{12,i}\}$ and $\{u_{9,i}, u_{12,i}\}$ for $C_{5,i}$, $\{u_{9,i}, u_{13,i}\}$ and $\{u_{9,i}, u_{14,i}\}$ for $C_{6,i}$, and $\{u_{3,i}, u_{11,i}\}$ and $\{u_{5,i}, u_{11,i}\}$ for $C_{7,i}$ – all connect vertices which only share this one coalition. Hence the (permanent) existence of such links does not make any other coalitions accessible. □

Discussion

In our model we assumed that, once a coalition is formed, all agents within the group can interact with each other. This is a natural approach when groups are not too big and the project requires some form of interaction among all agents. Let us also briefly discuss some other possible variants for locality and visibility in coalition formation games. For the first variant, we assume that it is more laborious to form a coalition than to maintain it. In particular, instead of a clique we assume that coalitions introduce an *organization graph* of temporary links into the network. Here we assume that the organization graph is a subgraph of the formation graph. For the second variant, we assume that agents not only connect to all other agents involved in their project but also get introduced to their friends (connected by permanent links). Formally, coalitions insert cliques in N, but nodes that are connected via hop-distance 2 can be used for visibility. This idea is inspired by properties of locally stable matching.

Organization Graphs

In many cases it is reasonable to assume that once a coalition is formed everyone within this coalition will interact with everybody else (e.g., via regular group meetings or events). However, for large coalitions, it is plausible that it takes more connectivity to bring together and arrange a group than to maintain it. Once the group is formed, a more sparse organization is sufficient to keep the group functional (e.g., a management hierarchy resulting in a tree structure). To incorporate this aspect, we keep the idea of a formation graph G to form some coalition C, but insert temporary links for C based on an *organization graph* G' with $E_{G'} \subseteq E_G$. Note that we assume the subgraph relation to hold for the actual mapping of G and G' to the vertices in N. In this setting, we are able to show that the positive results from Theorem 1 hold for all graphs G and $G' \subseteq G$.

Corollary 1. *Every local coalition formation game using formation graphs $\mathcal{G}$ and organization graphs $\mathcal{G}'$ with $E_{G'} \subseteq E_G$ for every $G \in \mathcal{G}$ and $G' \in \mathcal{G}'$ has a path to stability of length at most n using only local improvement steps from any starting state $\mathcal{S}$. Furthermore, random dynamics converge to a stable state in expected time $O(mn)$.*

The main insight is that the set of temporary links and accessible coalitions can only shrink. For the convergence times, we can then use exactly the same maximality argument as in Theorem 1. Observe that here we obtain polynomial-time convergence for all formation graphs under the condition that the visibility required for formation exceeds the subsequent interaction within the coalition. These results form an interesting first step to characterize convergence and stability in more general classes of games.

Triadic Closure

The exact definition of accessible coalition used by Arcaute and Vassilvitskii (2009) and Hoefer (2013) for locally stable matching slightly differs from the one proposed in our setting. In the case of matching, visibility of pairs is assumed via triadic closure, a central idea in the study of social networks. More formally, for a matching M, an edge

$e = \{u, v\} \notin M$ is a local blocking pair if e is a blocking pair and u and v are at distance at most 2 in the network $(V, L \cup M)$. Consequently, a new coalition (matching edge) can form based on a formation graph (length-2-path) involving an agent that is not part of the final coalition.

In the Appendix, we show how locally stable matching can be embedded in our setting of local coalition formation. Here we adopt the idea to our setting and discuss the consequences. Sadly, with visibility based on agents outside the coalitions, paths to stability must become exponential even for cliques or stars as organization graphs.

The direct generalization of visibility for locally stable matching to our setting is to add a clique for every existing coalition and then consider the triadic closure in the resulting graph as the basis for embedding the formation graph. Formally, in *local coalition formation games with triadic closure* coalition C is accessible in state $\mathcal{S}$ if there is some $G = (V_G, E_G) \in \mathcal{G}$ and some bijective map $\varphi : V_G \to C$ such that $\{u, v\} \in E_G \to dist(\varphi(u), \varphi(v), (V, L \cup \{(w, w')|w, w' \in C' \text{ for some } C' \in \mathcal{S}\})) \leq 2$.

Theorem 4. *Let $G = (V, E)$ be a star or a clique of at most 3 vertices. For every $n \in \mathbb{N}$ there is a local coalition formation game with triadic closure, n agents, $m = \Theta(n)$ coalitions, $\mathcal{G} = \{G\}$ and an initial state such that every sequence to a stable state requires $2^{\Theta(n)}$ improvement steps.*

Proof. We compose our example of a number of identical gadgets. Each gadget has a distinct start coalition and two distinct final coalitions. To establish one of the final coalitions the start coalition has to be formed and later on be destroyed. It is not possible to form both final coalitions through one creation of the start coalition. Thus, for both final coalitions to exist, the start coalition must have been formed twice. The gadgets will be connected such that the start coalition of gadget i can only be formed when both final coalitions of gadget $i - 1$ exist. Hence, to form both final coalitions of the k^{th} gadget we have to create the start coalition of the first gadget 2^k times. The construction is slightly different based on whether we consider $\mathcal{G}_1 = \{(\{u, v\}, \{\{u, v\}\}), (\{u, v, w\}, \{\{u, v\}, \{v, w\}\})\}$ the set of all stars with at most 3 vertices or $\mathcal{G}_2 = \{(\{u, v\}, \{\{u, v\}\}), (\{u, v, w\}, \{\{u, v\}, \{v, w\}, \{u, w\}\})\}$ the set of all cliques with at most 3 vertices.

Gadget i consists of $V_i = \{v_{1,i}, \ldots, v_{7,i}\}$, social links

$$L = \{\{v_{1,i}, v_{4,i}\}, \{v_{2,i}, v_{5,i}\}, \{v_{2,i}, v_{7,i}\}, \{v_{3,i}, v_{6,i}\}, \{v_{5,i}, v_{7,i}\}\}$$

plus $\{v_{1,i}, v_{6,i}\}$, if $\mathcal{G} = \mathcal{G}_2$, and potential coalitions $C_{1,i} = \{v_{1,i}, v_{2,i}, v_{3,i}\}$, $C_{2,i} = \{v_{2,i}, v_{4,i}\}$ $C_{3,i} = \{v_{3,i}, v_{5,i}\}$, $C_{4,i} = \{v_{2,i}, v_{6,i}\}$, and $C_{5,i} = \{v_{6,i}, v_{7,i}\}$. The benefits are given by $w(C_{1,i}) = 3i+1$, $w(C_{2,i}) = w(C_{4,i}) = 3i+2$, and $w(C_{3,i}) = w(C_{5,i}) = 3i + 3$. $C_{1,i}$ will play the role of our start coalition, and $C_{3,i}$ and $C_{5,i}$ will be the final coalitions of gadget i. To connect gadget i with gadget $i+1$ we identify vertex $v_{1,i+1}$ with $v_{4,i}$, vertex $v_{2,i+1}$ with $v_{7,i}$, and vertex $v_{3,i+1}$ with $v_{6,i}$. Further to make $C_{1,1}$ constantly accessible we add a vertex a and social links $\{a, v_{1,1}\}$, $\{a, v_{2,1}\}$, and $\{a, v_{3,1}\}$. As initial state we choose the empty coalition structure.

We will analyze the dynamics of gadget 1. All other gadgets i provide the same dynamics except that their start coalition is not constantly accessible but only when both final coalitions of their predecessor gadget $i - 1$ exist. Only then $v_{1,i}$ knows $v_{2,i}$ via $C_{3,i-1}$ and link $\{v_{5,i-1}, v_{7,i-1}\} = \{v_{5,i-1}, v_{2,i}\}$ and v_2 knows v_3 via $C_{5,i-1}$. Thus the star needed to make $C_{1,i}$ accessible in the case of $\mathcal{G} = \mathcal{G}_1$ $v_{1,i}$ exists. In the case of $\mathcal{G} = \mathcal{G}_2$ additionally $v_{1,i}$ and $v_{3,i}$ know each other (constantly) via $v_{1,i-1}$ which completes the clique. Now in the initial state gadget 1 is empty. Thus the only accessible coalition is $C_{1,1}$ which is always accessible due to the connections via a. Once $C_{1,1}$ is formed both $C_{2,1}$ and $C_{4,1}$ become accessible. As both coalitions are more valuable than $C_{1,1}$ one of them is formed in the next step. Let us assume that $C_{2,1}$ is formed. The dynamics for the case where $C_{4,1}$ forms first work analogously.The formation of $C_{2,1}$ destroys $C_{1,1}$ but makes $C_{3,1}$ accessible and thus a blocking coalition. When $C_{3,1}$ is formed (and $C_{2,1}$ discarded) $C_{1,1}$ becomes a blocking coalition again, as now all involved vertices are free. The formation of $C_{1,1}$ then makes $C_{4,1}$ accessible again and this time, because $C_{2,1}$ is blocked by $C_{3,1}$, $C_{4,1}$ forms the only blocking coalition. With $C_{4,1}$ formed $C_{5,1}$ becomes a blocking coalition. In the next step $C_{5,1}$ is formed, that is, both final coalitions exist and $C_{2,1}$ becomes available. Now the same dynamics kick off in gadget 2.

Note that we can alter the gadgets to only use coalitions of size 3 by adding a distinct vertex for each coalition of size 2 and connect it with one (for stars) or both (for cliques) vertices via a path of length 2 (using auxiliary vertices). □

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Appendix

Embedding of Locally Stable Matching

Lemma 2. *For any instance of locally stable matching with agents V_M, possible matching edges E_M, edge weights $w_M : E_M \to \mathbb{R}$ and permanent links N_M, there is a local coalition formation game which correctly maps the improvement dynamics of the matching instance.*

Proof. We have to deal with three different types of accessible edges: (1) $e \in E \cap N$, (2) $e = \{u, v\} \in E \setminus N$ such that there is some v' with $\{u, v'\}, \{v, v'\} \in N$, and (3) $e = \{u, v\} \in E \setminus N$ such that some edge $e' = \{u, v'\}$ is required to make e accessible via a social link $\{v, v'\} \in N$. We consider only edges of type (3) with $w(e) > w(e')$ as otherwise u would not want to deviate anyway.

We set $V = V_M \cup \{v_e \mid e \text{ of type (1) or (2)}\} \cup \{v_{e',e} \mid e \text{ of type (3)}\}$ and $N = \{\{u, v_e\}, \{v, v_e\} \mid e = \{u, v\} \text{ of type (1) or (2)}\} \cup \{\{v, v_{e',e}\}, \{v', v_{e',e}\} \mid e = \{u, v\} \text{ of type (3)}\}$. For the potential coalitions we have $\mathcal{C} = E_M \cup \{C_e = \{u, v, v_e\} \mid e \text{ of type (1) or (2)}\} \cup \{C_{e',e,1} = \{u, v, v', v_{e',e}\} \mid e \text{ of type (3)}\}$. The additional coalitions are auxiliary coalitions which will be used as intermediate step to simulate the resolution of a local blocking pair. Note, that these auxiliary coalitions are unique in the sense that there are no two combinations generating the same set.

For the coalitions appearing in the original matching game we keep the weights. For coalitions $C_{e',e} = \{u, v, v'\}$ as defined above finding the correct weight is slightly more involved. As we want to use $C_{e',e}$ as an intermediate step between e' and e, it has to be more valuable than e' but less than e. But those are not the only constraints. If v is interested in switching to u from whichever partner he currently has, we need $C_{e',e}$ to be attractive for v as well. To find suiting weights we order all values appearing in w_M in ascending order. For some weight $w_M(e)$ we define $succ(w_M(e))$ to be the direct predecessor of $w_M(e)$ in this order. Then we set $w(C_{e',e}) = succ(w_M(e)) + \frac{w_M(e) - succ(w_M(e))}{2}$ and $w(C_e) = succ(w_M(e)) + \frac{w_M(e) - succ(w_M(e))}{2}$. This way auxiliary coalitions are less valuable than the desired coalition but more valuable than every other coalition of lower weight. Further if the desired coalitions have the same value, then so do the auxiliary coalitions.

We use paths as formation graphs. Then we claim that starting from any coalition structure representing an valid matching in the original game, we can simulate the resolution of every local blocking pair using at most two improvement steps in the new game. Further there are no improvement steps in the coalition formation game which are not either the resolution of a local blocking pair or the intermediate step for the resolution of a local blocking pair. If some intermediate coalition is deleted by some coalition other than the one it was designed for, there is a sequence in the matching game which would have similar effects on the state.

First assume that for matching M edge $e = \{u, v\}$ is a blocking pair in the matching game. If u and v are in hop-distance at most 2 in N_M, there is some coalition $C_e = \{u, v, v_e\}$. By definition C_e is more worthy than any matching edge of u or v with value $< w_M(e)$. With the two newly introduced links connecting u and v with v_e as formation graph C_e is accessible. Thus C_e is a blocking coalition in the current state. Once C_e is formed, u and v are directly connected by a temporary link and can now deviate to the even more worthy coalition e. If the hop-distance between u and v is larger than 2, then there has to be some edge $e' \in M$ which makes e accessible. W.l.o.g. assume that $e' = \{u, v'\}$, and there is some social link $\{v, v'\}$ in N_M. Then $C_{e',e}$ is not only accessible via e', $\{v', v_{e',e}\}$, and $\{v, v_{e',e}\}$ but also more valuable than e' and any current coalition of v. Thus, again $C_{e',e}$ is a blocking coalition and once $C_{e',e}$ is formed, e becomes a blocking coalition.

Conversely, let $\mathcal{S}$ be a state of the coalition formation game representing a matching M and let C be a blocking coalition for $\mathcal{S}$. If $C_{e',e}$ for some $e = \{u, v\}, e' = \{u, v'\} \in E$, there must be some path connecting u, v, v', and $v_{e',e}$. As $e, e' \notin N$, one of them has to be part of the current state to provide a temporary link to complete the path. Because e is more valuable than $C_{e',e}$, $e' \in \mathcal{S}$. Then e would also be accessible and attractive for u in the matching game. Further v has to be single or in some coalition less valuable than $C_{e',e}$, that is, v is also interested in e. Thus e' is present in M, e is a blocking pair and the existence of auxiliary coalition $C_{e',e}$ as blocking coalition is correct. Similarly, if C_e is a blocking coalition, e is not only accessible in M but also more attractive for both agents than their current partners.

It remains to discuss the case where instead of forming the coalition e from $C_{e',e}$ respectively C_e the auxiliary coalition C gets deleted by some other coalition C'. This can only happen if C' is strictly more valuable than C and thus at least as valuable as the matching edge resulting from C. In fact, C' has to be strictly more valuable than e. This is due to the fact, that before forming some edge $e' \in E$ first the suiting auxiliary coalition which involves all agents of e' has to be formed. Thus, if C' would be some edge $e' \in E$, C could not exists as the auxiliary coalition for e' already included all agents of e'. On the other hand, if C' is the auxiliary coalition for some e' with $w(e) = w(e')$ then $w(C) = w(C')$, that is, C' cannot be a blocking coalition if C exists. Thus, C' has to be the auxiliary coalition for some e' with $w(e') > w(e)$. We can then interpret the situation as e already being formed and then deleted by e'. $\square$